Differential Operator Endomorphisms of an Euler-Lagrange Complex

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Abstract. The main results of our paper deal with the lifting problem for multilinear differential operators between complexes of horizontal de Rham forms on the infinite jet bundle. We answer the question when does an n-multilinear differential operator from the space of $(N,0)$-forms (where N is the dimension of the base) to the space of $(N-s,0)$-forms allow an n-multilinear extension of degree $(-s,0)$ defined on the whole horizontal de Rham complex. To study this problem we define a differential graded operad DEnd_* of multilinear differential endomorphisms, which we prove (Theorem 4.8) to be acyclic in positive degrees (negative mapping degrees) and describe the cohomology group in degree zero in terms of the characteristic (Definition 4.3). Corollary 4.9 uses this result to solve the lifting problem. An important application to mathematical physics is the proof of existence of a strongly homotopy Lie algebra structure extending a Lie bracket on the space of functionals (Theorem 6.7).
The results of the paper were announced in [14].

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1. Introduction.

Our interest in the subject began when we read the paper [2] in which the authors construct a strong homotopy Lie algebra extending the Poisson bracket on local functionals. The horizontal de Rham complex on the infinite jet bundle (one row of the variational bicomplex) can be augmented over the space of local functionals by a map defined as integration of the pull-back of an $(N,0)$-form by a section of the jet bundle, see Definition 6.1. This defines a projective complex which, after passing to a quotient by the space of constants in the

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bidegree $(0,0)$ term, gives a resolution of the space of local functionals. One can define an extension of the Poisson bracket by applying standard techniques for resolutions (modulo constants). The problem with this approach is that the extension is done value by value with no control of the type of operator (differential, continuous, etc.) being defined; when the Poisson bracket is given by a differential operator, the higher brackets may not be, and, in fact, need not have any particular regularity properties.

We use a different approach for which all extensions will belong to a natural class of differential operators, "local differential operators," which is the one to which the Poisson brackets are usually assumed to belong and includes the Euler-Lagrange operator and the total horizontal derivatives, see 3.1 for the definition and explanation of the terminology.

The precise statements of our results are both technically and notationally complicated, and therefore, we have decided to motivate the reader by some 'toy models' – the case of linear (i.e. not multilinear) operators on the zero-dimensional bundle $\mathbf{R}^N \to \mathbf{R}^N$, where the forms on the jet bundle are, of course, ordinary de Rham forms on $\mathbf{R}^N$. In Section 3 we add vertical variables, but still remain in the linear case. The multilinear situation is introduced in Section 4.

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2. First toys.

Let us consider the de Rham complex $\Omega^0(\mathbf{R}) \xrightarrow{d} \Omega^1(\mathbf{R})$ on the one-dimensional Euclidean space $\mathbf{R}$. The space, $\mathrm{DO}(\mathbf{R})$, of linear differential operators on $\mathbf{R}$, consists of maps

$$\begin{aligned} A &= \sum_{i\geq 0} a_i \left(\frac{d}{dx}\right)^i : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}), \\ A(f) &= \sum_{i\geq 0} a_i d^i f/dx^i, \text{ for } f \in C^\infty(\mathbf{R}), \end{aligned}$$

where $a_i = a_i(x)$, $i \geq 0$, is a sequence of smooth functions such that $a_i = 0$ for i sufficiently large. Any differential operator A has a natural extension to the space of one forms, $A_1 : \Omega^1(\mathbf{R}) \to \Omega^1(\mathbf{R})$, given by

$$A_1(f dx) := A(f) dx \in \Omega^1(\mathbf{R}). \tag{1}$$

We are looking for a differential operator $A_0 : \Omega^0(\mathbf{R}) \to \Omega^0(\mathbf{R})$, $A_0(g) = \sum_{j\geq 0} b_j d^j g/dx^j$, which lifts A_1 in the sense that $dA_0 = A_1 d$ or, diagrammatically,

(2)
$$\begin{tikzcd} \Omega^0(\mathbf{R}) \arrow[r, "d"] \arrow[d, "A_0"'] & \Omega^1(\mathbf{R}) \arrow[d, "A_1"] \\ \Omega^0(\mathbf{R}) \arrow[r, "d"] & \Omega^1(\mathbf{R}) \end{tikzcd}$$

The condition $dA_0 = A_1 d$ can be expanded into the system

$$
\begin{aligned}
0 &= \frac{db_0}{dx} \\
a_0 &= b_0 + \frac{db_1}{dx} \\
&\vdots \\
a_{k-1} &= b_{k-1} + \frac{db_k}{dx},\ k \geq 1.
\end{aligned} \tag{3}
$$

Assuming that A_1 has order n and A_0 has finite order, we get the solution

$$
\begin{aligned}
b_n &= a_n \\
&\vdots \\
b_{n-k} &= a_{n-k} + \sum_{1 \leq j \leq k} (-1)^j \frac{d^j a_{n-k+j}}{dx^j},\ 1 \leq k \leq n.
\end{aligned}
$$

$$
0 = \sum_{1 \leq j \leq n+1} (-1)^j \frac{d^j a_{j-1}}{dx^j}. \tag{4}
$$

Equation (4) imposes the only restriction on the operators, i.e. on the coefficients a_i and b_j, for which a lifting exists in the context of our toy model. Appropriate generalizations of this condition will reappear in all subsequent examples. The following definitions will allow a precise formulation of necessary and sufficient conditions for a lifting.

Definition 2.1. *The* formal adjoint *of the differential operator*

$$A = \sum_{i \geq 0} a_i \left(\frac{d}{dx}\right)^i$$

is defined as the differential operator

$$A^+ := \sum_{i \geq 0} (-1)^i \left(\frac{d}{dx}\right)^i \circ a_i,$$

where $(d/dx)^i \circ a_i$ is the composition of the operator of the i-th derivative with multiplication by a_i. The characteristic *of A is the function $\chi : \mathrm{DO}(\mathbf{R}) \longrightarrow C^\infty(\mathbf{R})$ given by*

$$\chi(A) := A^+(1) = \sum_{i \geq 0} (-1)^i \frac{d^i a_i}{dx^i} \in C^\infty(\mathbf{R}),$$

where 1 denotes the constant function.

As a consequence of the equation $(AB)^+ = B^+ A^+$ for adjoints, the characteristic satisfies

$$\chi(A \circ B) = B^+(\chi(A)), \tag{5}$$

see [16]. As an immediate consequence of (5), we obtain

$$\chi(A \circ \frac{d}{dx}) = -\frac{d}{dx}\chi(A) \quad \text{and} \quad \chi(\frac{d}{dx} \circ A) = 0. \tag{6}$$

An important, though obvious, property is that χ is a projector, $\chi^2 = \chi$.

Proposition 2.2. *For an arbitrary differential operator $A : \Omega^1(\mathbf{R}) \to \Omega^1(\mathbf{R}) \cong C^\infty(\mathbf{R})$, there exists a differential operator $\tilde{A} : \Omega^1(\mathbf{R}) \to \Omega^0(\mathbf{R}) \cong C^\infty(\mathbf{R})$ such that*

$$A = d \circ \tilde{A} + \chi(A). \tag{7}$$

Proof. Clearly it is enough to prove the proposition for $A(f) = (a_n d^n f/dx^n)dx$, for an arbitrary $n \geq 0$. For such A, (7) is satisfied with

$$\tilde{A}(f) := \sum_{0 \leq i < n} (-1)^i \frac{d^i a_n}{dx^i} \cdot \frac{d^{n-i-1} f}{dx^{n-i-1}}.$$

■

Corollary 2.3. *In the situation of (2), the lift A_0 of the operator A_1 exists if and only if*

$$\chi(A_1) = \mathit{constant}.$$

Proof. If $A_1 d = dA_0$ for some A_0, applying Proposition 2.2 to $A = A_1$ and composing on the right with d gives

$$d \circ A_0 = A_1 \circ d = (d \circ \tilde{A} + \chi(A_1)) \circ d = d \circ \tilde{A} \circ d + \chi(A_1) \circ d.$$

From the last equation, the projector property of χ and (6), we have

$$-\frac{d}{dx}(\chi(A_1)) = \chi(\chi(A_1) \circ d) = \chi(d \circ (A_0 - \tilde{A} \circ d)) = 0,$$

so $\chi(A_1) = $ constant. On the other hand, if the assumption of the corollary is true, we can put $A_0 := \tilde{A} \circ d + \chi(A_1)$ and $d \circ A_0 = A_1 \circ d$. ■

Let us move on to a higher-dimensional version of the above situation. The following notation is standard:

$$\left(\frac{\partial}{\partial \mathbf{x}}\right)^I := \left(\frac{\partial}{\partial x^1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x^N}\right)^{i_N}, \tag{8}$$

where $I = (i_1, \ldots, i_N)$, $i_1, \ldots, i_N \geq 0$.

Definition 2.4. *Given a linear (partial) differential operator*

$$A = \sum_I a_I \left(\frac{\partial}{\partial \mathbf{x}}\right)^I \tag{9}$$

where $a_I = a_I(\mathbf{x})$ are smooth functions on $\mathbf{R}^N$ and

$$a_I \neq 0 \text{ only for finitely many indices } I, \tag{10}$$

we define the characteristic *to be the function*

$$\chi(A) := \sum_I (-1)^I \left(\frac{\partial}{\partial \mathbf{x}}\right)^I a_I \in C^\infty(\mathbf{R}^N), \tag{11}$$

where $(-1)^I := (-1)^{i_1+\cdots+i_N}$.

There is a natural definition of a linear partial differential operator on the space of de Rham forms. The space $\Omega^k(\mathbf{R}^N)$ is a free $C^\infty(\mathbf{R}^N)$-module with basis

$$\{(d\mathbf{x})^\epsilon := (dx^1)^{\epsilon^1} \wedge \cdots \wedge (dx^N)^{\epsilon^N};\ \epsilon = (\epsilon^1, \dots, \epsilon^N),\ \epsilon^1, \dots, \epsilon^N \in \{0,1\},\ |\epsilon| := \sum \epsilon^i = k\}.$$

Definition 2.5. *A linear map $A : \Omega^k(\mathbf{R}^N) \to \Omega^l(\mathbf{R}^N)$ is a differential operator if in the expansion*

$$\sum_\epsilon A(f_\epsilon (d\mathbf{x})^\epsilon) = \sum_{\epsilon,\delta} A^\epsilon_\delta(f_\epsilon)(d\mathbf{x})^\delta,$$

the 'matrix elements' A^ϵ_δ are differential operators in the sense of Definition 2.4.

Let us consider the complex of de Rham forms on $\mathbf{R}^N$, $N \geq 1$:

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^0(\mathbf{R}^N) \xrightarrow{d} \Omega^1(\mathbf{R}^N) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{N-1}(\mathbf{R}^N) \xrightarrow{d} \Omega^N(\mathbf{R}^N) \longrightarrow 0$$

and a differential operator $A : \Omega^N(\mathbf{R}^N) \to \Omega^N(\mathbf{R}^N) \cong C^\infty(\mathbf{R}^N)$. The following statement generalizes Proposition 2.2 and is proved by an easy induction on the number of variables.

Proposition 2.6. *For any differential operator $A : \Omega^N(\mathbf{R}^N) \to \Omega^N(\mathbf{R}^N)$ as defined above, there exists a differential operator $\tilde{A} : \Omega^N(\mathbf{R}^N) \to \Omega^{N-1}(\mathbf{R}^N)$ such that*

$$A = d \circ \tilde{A} + \chi(A).$$

Corollary 2.7. *A differential operator $A_N : \Omega^N(\mathbf{R}^N) \to \Omega^N(\mathbf{R}^N)$ can be lifted to a sequence of differential operators $\{A_s : \Omega^s(\mathbf{R}^N) \to \Omega^s(\mathbf{R}^N)\}_{0 \leq s \leq N}$ such that $dA_s = A_{s+1}d$ if and only if*

$$\chi(A_N) = \text{constant}.$$

In this case, the lift can be chosen in such a way that

$$A_s = \chi(A_N) \text{ (multiplication by } \chi(A_N)), \tag{12}$$

for $0 \leq s \leq N-2$.

Proof. Our situation is described by the following diagram:

$$\begin{tikzcd}
\Omega^0(\mathbf{R}^N) \arrow[r, "d"] \arrow[d, dashed, "A_0"] & \cdots \arrow[r, "d"] & \Omega^{N-1}(\mathbf{R}^N) \arrow[r, "d"] \arrow[d, dashed, "A_{N-1}"] & \Omega^N(\mathbf{R}^N) \arrow[d, "A_N"] \\
\Omega^0(\mathbf{R}^N) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{N-1}(\mathbf{R}^N) \arrow[r, "d"] & \Omega^N(\mathbf{R}^N)
\end{tikzcd}$$

As in the proof of Corollary 2.3, we apply Proposition 2.6 to A_N. Then for

$$d^{(i)} := d|_{S^{(i)}} \text{ for } S^{(i)} := \mathrm{Span}_{C^\infty(\mathbf{R}^N)}(dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n) \subset \Omega^{N-1}(\mathbf{R}^N), 1 \leq i \leq N,$$

we have

$$A_N \circ d^{(i)} = d \circ \tilde{A} \circ d^{(i)} + \chi(A_N) \circ d^{(i)}.$$

As before, we conclude that if the lift A_{N-1} exists, then

$$0 = \chi(d \circ A_{N-1}|_{S^{(i)}}) = \chi(d \circ \tilde{A} \circ d^{(i)}) + \chi(\chi(A_N) \circ d^{(i)}) = \frac{\partial \chi(A_N)}{\partial x_i}$$

for each i, so $\chi(A_N)$ is constant.

If $\chi(A_N)$ is constant, then setting $A_{N-1} := \tilde{A}d + \chi(A_N)$ and $A_s := \chi(A_N)$, for $s \leq N-2$, defines a lift of A_N with the desired property (12). ■

3. More serious models.

Let $E \to M$ be a smooth vector bundle over a smooth manifold M. We will, in fact, always suppose that $M = \mathbf{R}^N$, with coordinates $\mathbf{x} = (x^1, \ldots, x^N)$, and that E is the trivial one-dimensional bundle, $E = \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}^N$, with only one 'vertical' coordinate u. More vertical coordinates present only notational difficulties and all our results directly generalize to this situation. The case of a general manifold M and a possibly nontrivial bundle E can be studied by standard globalization techniques, where our situation will serve as the local model.

We will consider forms and functions on the infinite jet bundle $J^\infty E$ over E. This jet bundle has coordinates $(\mathbf{x}, \mathbf{u}) = (x^i, u_J)$, where $1 \leq i \leq N$, J runs over all multi-indices $(j_1, \ldots, j_N)$, $j_1, \ldots, j_N \geq 0$, and u_J is the coordinate such that

$$u_J(j^\infty(\phi)) = \left(\frac{\partial}{\partial \mathbf{x}}\right)^J \phi,$$

see (8) for the notation. The order of J is defined as $|J| = j_1 + \cdots + j_N$.

Recall that a *local function*, $f = f(x^i, u_J)$, is by definition the pullback of a smooth function on some $J^k E$, and thus depends only on finitely many u_J's. We denote by $\mathrm{Loc}(E)$ the vector space of all local functions. Let $(\Omega^*(J^\infty E), d)$ be the complex of de Rham forms on $J^\infty E$ whose coefficients are local functions. It is well-known [1] that the differential on $\Omega^*(J^\infty E)$ decomposes into a horizontal and a vertical component $d = d_H + d_V$, defining the structure of a bicomplex, the so-called *variational bicomplex*,

$$\Omega^*(J^\infty E) = \bigoplus_{k+l=*,\ k,l\geq 0} \Omega^{k,l}(J^\infty E);\ d = d_H + d_V.$$

Let us denote by d/dx^i, $1 \leq i \leq N$, the total derivative with respect to x^i,

$$\frac{d}{dx^i} := \frac{\partial}{\partial x^i} + \sum_J u_{iJ} \frac{\partial}{\partial u_J},$$

where

$$iJ = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_N). \tag{13}$$

Given that the i-th slot of the multi-index J indicates the number of x_i derivatives, it would make more sense to denote one more x_i derivative on u_J by $u_{J+\delta_i}$. Our convention is a (perhaps futile) attempt to simplify an increasingly complicated system of notation. Let us remark also that d/dx^i is usually denoted by D_i.

Then the 'horizontal' differential $d_H : \Omega^{k,*}(J^\infty E) \to \Omega^{k+1,*}(J^\infty E)$ is given by the formula

$$d_H \omega = \sum_{1\leq i\leq N} dx^i \wedge \frac{d}{dx^i}\omega.$$

As in (8) we denote

$$\left(\frac{d}{d\mathbf{x}}\right)^I = \left(\frac{d}{dx^1}\right)^{i_1} \cdots \left(\frac{d}{dx^N}\right)^{i_N},\ \text{for } I = (i_1, \ldots, i_N).$$

In order to deal with vertical derivatives, we introduce the expression

$$\left(\frac{\partial}{\partial \mathbf{u}}\right)^\alpha = \prod_J \frac{\partial^{\alpha(J)}}{\partial u_J^{\alpha(J)}},$$

where α is a non-negative integer valued function on the multi-indices J and $\alpha(J) \neq 0$ for only finitely many J.

The following definitions are crucial.

Definition 3.1. *The formal differential degree of* α*, denoted* $\deg_f(\alpha)$*, is the maximal order of multi-index* J *such that* $\alpha(J) \neq 0$*. We say that a linear map* $A : \mathrm{Loc}(E) \to \mathrm{Loc}(E)$ *is a LDO (local differential operator) if it is of the form*

$$A(f) = \sum_{I,\alpha} p_{I,\alpha}(\mathbf{x}, \mathbf{u}) \left(\frac{d}{d\mathbf{x}}\right)^I \left(\frac{\partial}{\partial \mathbf{u}}\right)^\alpha (f),\ \ f \in \mathrm{Loc}(E),$$

where $p_{I,\alpha}(\mathbf{x}, \mathbf{u}) \in \mathrm{Loc}(E)$*, and has the property that*

(14) *for each integer* n*, there are only finitely many* $p_{I,\alpha} \neq 0$ *with* $\deg_f(\alpha) \leq n$

Condition (14) guarantees that $A(f)$ is a finite sum for each $f \in \mathrm{Loc}(E)$. The term 'local differential operator' expresses the fact that these operators preserve the space of local functions.

It will be useful to introduce the total symbol of a LDO, using variables ξ^i, η_J to represent $d/dx^i, \partial/\partial u_J$ respectively. For a LDO A, as defined above, we have

$$\sigma(A) = \sum_{I,\alpha} p_{I,\alpha}(\mathbf{x}, \mathbf{u})(\xi)^I(\eta)^\alpha$$

For a LDO A we define the *characteristic* to be the *differential operator* (in contrast with the case of no vertical variable, where it was a function)

$$\chi(A) := \sum_{I,\alpha}(-1)^I \left(\left(\frac{d}{d\mathbf{x}}\right)^I p_{I,\alpha}(\mathbf{x}, \mathbf{u})\right)\left(\frac{\partial}{\partial \mathbf{u}}\right)^\alpha \in \mathrm{LDO}. \tag{15}$$

Observe that all the horizontal derivatives appear only in the coefficients, $(d/d\mathbf{x})^I p_{I,\alpha}(\mathbf{x}, \mathbf{u})$, so as an operator $\chi(A)$ contains only vertical derivatives. We will be interested in the lifting problem for the bottom row of the variational bicomplex:

$$\Omega^{0,0}(J^\infty E) \xrightarrow{d_H} \Omega^{1,0}(J^\infty E) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{N-1,0}(J^\infty E) \xrightarrow{d_H} \Omega^{N,0}(J^\infty E). \tag{16}$$

which is the initial segment of the *Euler-Lagrange complex* [1].

We extend Definition 3.1 to maps of forms as in Definition 2.5. Namely, a local differential operator $A : \Omega^{k,0}(J^\infty E) \to \Omega^{l,0}(J^\infty E)$ is a linear map whose 'matrix coefficients' are LDOs in the sense of Definition 3.1.

Then $f \mapsto f \cdot dx^1 \wedge \cdots \wedge dx^N$ gives an identification $\mathrm{Loc}(E) \cong \Omega^{N,0}(J^\infty E)$. Formula (15) thus defines the characteristic also for a LDO $A : \Omega^{N,0}(J^\infty E) \to \Omega^{N,0}(J^\infty E)$. The following Proposition is an analog of Proposition 2.6.

Proposition 3.2. *For each LDO $A : \Omega^{N,0}(J^\infty E) \to \Omega^{N,0}(J^\infty E)$, there exists a LDO $\tilde{A} : \Omega^{N,0}(J^\infty E) \to \Omega^{N-1,0}(J^\infty E)$ such that*

$$A = d_H \tilde{A} + \chi(A).$$

Corollary 3.3. *A LDO $A_N : \Omega^{N,0}(J^\infty E) \to \Omega^{N,0}(J^\infty E)$ can be lifted into a sequence $\{A_s : \Omega^{s,0}(J^\infty E) \to \Omega^{s,0}(J^\infty E)\}_{0 \le s \le N}$ of LDO's if and only if*

$$\chi(A_N \frac{d}{dx^i}) = 0, \ 1 \le i \le N. \tag{17}$$

In this case the lift can be chosen in such a way that

$$A_s = \chi(A_N), \ \textit{for } 0 \le s \le N-2. \tag{18}$$

Here (18) means that A_s acts 'diagonally' by $A_s(f(d\mathbf{x})^\epsilon) = \chi(A_N)(f(d\mathbf{x})^\epsilon)$. The corollary will be a consequence of more general statements of Section 4.

We end this section with some calculations useful in the sequel. First, the commutation relation between $\partial/\partial u_J$ and d/dx^i is given by:

$$\frac{\partial}{\partial u_J}\frac{d}{dx^i} - \frac{d}{dx^i}\frac{\partial}{\partial u_J} = \begin{cases} 0, & \text{for } j_i = 0, \text{ and} \\ \partial/\partial u_K, & \text{for } J = iK. \end{cases} \tag{19}$$

It will be convenient to write $\eta_{J/i}$ for η_K when $J = iK$ and, if i does not appear in J, then $\eta_{J/i} = 0$. Relations (19) can be written very compactly in terms of symbols. Defining the operator

$$\Theta^i = \sum_J \eta_{J/i}\frac{\partial}{\partial \eta_J}, \tag{20}$$

acting as a derivation on symbols. In the symbol calculus, commutation relation (19) becomes

$$\eta_J\xi^i - \xi^i\eta_J = \Theta^i(\eta_J). \tag{21}$$

Thus for any monomial η^α,

$$\eta^\alpha\xi^i - \xi^i\eta^\alpha = \Theta^i(\eta^\alpha).$$

For reference, we state another commutation relation in the symbol calculus which we will need later:

$$\xi^i p(\mathbf{x},\mathbf{u}) = \frac{d}{dx^i}p(\mathbf{x},\mathbf{u}) + p(\mathbf{x},\mathbf{u})\xi^i. \tag{22}$$

Using relations (21) and (22) we easily deduce that,

$$\sigma(A\frac{d}{dx^i} - \frac{d}{dx^i}A) = (\Theta^i - \frac{d}{dx^i})\sigma(A), \text{ for } A \in \text{LDO}. \tag{23}$$

The lemma follows immediately from the formula above.

Define a 'diagonal LDO' $A : \Omega^{k,0}(J^\infty E) \to \Omega^{k,0}(J^\infty E)$ to be to be an operator of the form $A(f(d\mathbf{x})^\epsilon) = A(f)(d\mathbf{x})^\epsilon$, where A is a LDO as above.

Lemma 3.4. *A 'diagonal LDO' A commutes with the horizontal differential d_H if and only if*

$$(\Theta^i - \frac{d}{dx^i})\sigma(A) = 0, \text{ for all } 1 \le i \le N.$$

Finally observe that

$$\sigma(\chi(A\frac{d}{dx^i})) = (\Theta^i - \frac{d}{dx^i})\sigma(\chi(A)), \text{ for all } 1 \le i \le N.$$

Thus, the assumption (17) implies that the operators $A_s = \chi(A_N)$ of (18) commute with the differentials. Explicitly:

Proposition 3.5. *Suppose A is a LDO as above and that $\chi(A\, d/dx^i) = 0$, for $1 \le i \le N$. Then the diagonal operator $\chi(A)$ commutes with the differential d_H.*

4. Main results.

Let us consider, as in Section 3, the infinite jets on the one-dimensional trivial vector bundle over $\mathbf{R}^N$. First, we need to introduce multilinear differential operators.

Definition 4.1. *An* n-multilinear LDO *(local differential operator) is an* n*-linear map* $A : \mathrm{Loc}(E)^{\otimes n} \to \mathrm{Loc}(E)$ *of the form*

$$A(f_1, \dots, f_n) = \sum p_{I_1,\dots,I_n,\alpha_1,\dots,\alpha_n} \left(\left(\frac{d}{d\mathbf{x}}\right)^{I_1} \left(\frac{\partial}{\partial \mathbf{u}}\right)^{\alpha_1} f_1 \right) \cdots \left(\left(\frac{d}{d\mathbf{x}}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}}\right)^{\alpha_n} f_n \right), \tag{24}$$

where $f_1, \dots, f_n \in \mathrm{Loc}(E)$ *and* $p_{I_1,\dots,I_n,\alpha_1,\dots,\alpha_n} = p_{I_1,\dots,I_n,\alpha_1,\dots,\alpha_n}(\mathbf{x}, \mathbf{u}) \in \mathrm{Loc}(E)$ *are local functions. We also require that, for any* m*, there are only finitely many multi-indices,* $I_1, \dots, I_n; \alpha_1, \dots, \alpha_n$*, such that*

$$\sum_{1 \leq i \leq n} \deg_f(\alpha_i) \leq m \text{ and } p_{I_1,\dots,I_n,\alpha_1,\dots,\alpha_n} \neq 0. \tag{25}$$

Condition (25) which is the analog of (14) guarantees that the operator has well defined values on n-tuples of local functions.

If we denote by $d/d\mathbf{x}_j$ (resp $\partial/\partial \mathbf{u}_j$) the total derivative (resp. the partial derivative) acting on the j-th function, $1 \leq j \leq n$, then we can write the operator in (24) in a more concise form as

$$A = \sum p_{I_1,\dots,I_n,\alpha_1,\dots,\alpha_n} \left(\frac{d}{d\mathbf{x}_1}\right)^{I_1} \cdots \left(\frac{d}{d\mathbf{x}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n}.$$

We denote the vector space of all such n-linear local differential operators by $\mathrm{LDO}(n)$.

Proposition 4.2. *The collection* $\mathrm{LDO} = \{\mathrm{LDO}(n)\}_{n \geq 1}$ *with the composition maps*

$$\gamma : \mathrm{LDO}(l) \otimes \mathrm{LDO}(k_1) \otimes \cdots \otimes \mathrm{LDO}(k_1) \to \mathrm{LDO}(k_1 + \cdots + k_l)$$

given by $\gamma(A; A_1, \dots, A_l) := A(A_1, \dots, A_l)$ *and the action of the symmetric group given by* $\sigma A(f_1, \dots, f_n) := A(f_{\sigma^{-1}(1)}, \dots, f_{\sigma^{-1}(n)})$*,* $\sigma \in \Sigma_n$*, forms an operad.*

Proof. The claim is almost obvious. The only thing which has to be verified is that the composition $A(A_1, \dots, A_l)$ is again a local differential operator. But the commutation relation (19) says how to move the total derivatives $d/d\mathbf{x}$ over the horizontal derivatives $\partial/\partial u_J$ to the left, which enables us to write the composition $A(A_1, \dots, A_l)$ in the form (24). ■

Define the change of variables,

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \longmapsto (\mathbf{y}_1 := \mathbf{x}_1, \mathbf{y}_2 := \mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{y}_n := \mathbf{x}_n - \mathbf{x}_1),$$

that is, $y_1^i = x_1^i$, and $y_j^i = x_j^i - x_1^i$ for $2 \le j \le n$ and $1 \le i \le N$. We have

$$\frac{d}{d\mathbf{y}_1} := \frac{d}{d\mathbf{x}_1} + \cdots + \frac{d}{d\mathbf{x}_n} \quad \text{and} \quad \frac{d}{d\mathbf{y}_j} := \frac{d}{d\mathbf{x}_j}, \text{ for } 2 \le j \le n. \tag{26}$$

Then such an operator, A, can be written in the *polarized form* (with different indexing) as

$$\sum q_{I_1,I_2,\ldots,I_n;\alpha_1,\ldots,\alpha_n} \left(\frac{d}{d\mathbf{y}_1}\right)^{I_1} \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n}. \tag{27}$$

Definition 4.3. *For $A \in \mathrm{LDO}(n)$ in the polarized form of (27), the* characteristic $\chi(A) \in \mathrm{LDO}(n)$ *is defined as*

$$\chi(A) = \sum (-1)^{I_1} \left(\left(\frac{d}{d\mathbf{x}}\right)^{I_1} q_{I_1,I_2,\ldots,I_n;\alpha_1,\ldots,\alpha_n} \right) \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n}.$$

Remark. It was necessary to introduce the polarized form before defining the characteristic so that composition with d/dx^i would introduce only one new horizontal derivative, d/dy_1^i, or equivalently, integration by parts on the output of the multilinear operator would affect only one tensor component. The relevant formula is

$$\begin{aligned} \frac{d}{dx^i} \circ A \; &= \; \sum \frac{dq_{I_1,I_2,\ldots,I_n;\alpha_1,\ldots,\alpha_n}}{dx^i} \left(\frac{d}{d\mathbf{y}_1}\right)^{I_1} \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n} \\ &+ \; \sum q_{I_1,I_2,\ldots,I_n;\alpha_1,\ldots,\alpha_n} \left(\frac{d}{d\mathbf{y}_1}\right)^{iI_1} \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n}. \end{aligned} \tag{28}$$

Let $\mathrm{LDO}^0(n)$ denote the image of $\chi : \mathrm{LDO}(n) \to \mathrm{LDO}(n)$. It consists of those $A \in \mathrm{LDO}(n)$ whose polarized form does not contain total derivatives $d/d\mathbf{y}_1$. The endomorphism χ is a projection onto $\mathrm{LDO}^0(n)$, $\chi^2 = \chi$, and

$$\mathrm{LDO}^0(n) \cong \frac{\mathrm{LDO}(n)}{\{A \in \mathrm{LDO}(n);\; \chi(A) = 0\}}. \tag{29}$$

To simplify the degree conventions, we regrade the horizontal complex $\Omega^{*,0}(J^\infty E)$ by introducing

$$\Omega_i(J^\infty E) := \Omega^{N-i,0}(J^\infty E),\; 0 \le i \le N. \tag{30}$$

Thus $(\Omega_*(J^\infty E), d_H)$ is now a chain complex, $\deg(d_H) = -1$.

As we deal with increasingly complicated situations, the definition of the characteristic becomes more complicated, but once again, given $A : [(\Omega_*(J^\infty E))^{\otimes n}]_0 \to \Omega_0(J^\infty E)$, there exists a $\tilde{A} : [(\Omega_*(J^\infty E))^{\otimes n}]_0 \to \Omega_1(J^\infty E)$ such that

$$A = d_H \tilde{A} + \chi(A). \tag{31}$$

Let us introduce the differential graded operad $\mathrm{DEnd}_* = \{\mathrm{DEnd}_*(n)\}_{n\geq 1}$ of local differential operator endomorphisms of the (regraded) horizontal de Rham complex $\Omega_*(J^\infty E)$. This means that $\mathrm{DEnd}_k(n)$ consists of degree k graded vector space maps $f : (\Omega_*(J^\infty E))^{\otimes n} \to \Omega_*(J^\infty E)$ with 'matrix coefficients' from $\mathrm{LDO}(n)$. An element of $\mathrm{DEnd}_k(n)$ is thus a sequence $f = \{f_s\}$, with

$$f_s : [(\Omega_*(J^\infty E))^{\otimes n}]_s \to \Omega_{s+k}(J^\infty E). \tag{32}$$

Observe that f_s may be nonzero only for $\max(-k, 0) \leq s \leq \min(Nn, N-k)$. The differential δ on DEnd_* is given by the usual formula

$$(\delta f)_s := d_H f_s - (-1)^{\deg(f)} f_{s-1} d_H^{\otimes n}, \tag{33}$$

where $d_H^{\otimes n}$ is the standard extension of d_H to the tensor product. Thus $\delta f = 0$ if and only if f is a chain map. The composition maps and the action of the symmetric group are given as in Proposition 4.2; the arguments that this indeed defines an operad structure are the same.

In this context, the lifting problem analogous to the one discussed in the previous sections can be formulated as an extending a LDO, $f_0 : [(\Omega_*(J^\infty E))^{\otimes n}]_0 \to \Omega_0(J^\infty E)$, to a cocycle in $\mathrm{DEnd}_0(n)$. Observe that $[\Omega_*(J^\infty E)^{\otimes n}]_0$ consists of elements of the form

$$(\omega_1 dx^1 \wedge \cdots \wedge dx^N) \otimes \cdots \otimes (\omega_n dx^1 \wedge \cdots \wedge dx^N),\ \omega_i \in \mathrm{Loc}(E),\ 1 \leq i \leq n,$$

Since both $\Omega_0(J^\infty E)$ and $[\Omega^*(J^\infty E)^{\otimes n}]_0$ are rank one modules over $\mathrm{Loc}(E)$, $\Omega_0(J^\infty E) \cong \mathrm{Loc}(E)$ and $[\Omega^*(J^\infty E)^n]_0 \cong \mathrm{Loc}(E)^{\otimes n}$, so f_0 can be interpreted as an element of $\mathrm{LDO}(n)$. The space $\Omega_1(J^\infty E)$ consists of elements $\sum \omega_i(dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^N)$, and thus is a rank n module, isomorphic to $\mathrm{Loc}(E)^{\oplus N}$.

From the definition,

$$[(\Omega_*(J^\infty E))^{\otimes n}]_1 = \bigoplus_{1\leq j\leq n} \Omega_0(J^\infty E) \otimes \cdots \otimes \Omega_1(J^\infty E) \otimes \cdots \otimes \Omega_0(J^\infty E)$$

($\Omega_1(J^\infty E)$ at the jth position). We derive the identification

$$[(\Omega_*(J^\infty E))^{\otimes n}]_1 = \bigoplus_{\substack{1\leq i\leq N\\ 1\leq j\leq n}} \mathrm{Loc}(E)^{\otimes n}_{i,j} \cong (\mathrm{Loc}(E)^{\otimes n})^{\oplus Nn}. \tag{34}$$

Then d/dx^i_j represents the boundary operator on the component of $[(\Omega_*(J^\infty E))^{\otimes n}]_1$ corresponding to $\mathrm{Loc}(E)^{\otimes n}_{i,j}$.

As before, our strategy is to invoke (31) to find $\tilde{f}_0 : [(\Omega_*(J^\infty E))^{\otimes n}]_0 \to \Omega_1(J^\infty E)$, to construct f_1, the first stage of the extension, using $\tilde{f}_0$, and continue from there.

The identity $\chi(d/dx^i \circ A) = 0$ follows from the definition of the characteristic. Right composition of both sides of equation (31) for the operator f_0 with d/dx^i_j gives

$$f_0 \frac{d}{dx^i_j} = d_H \tilde{f}_0 \frac{d}{dx^i_j} + \chi(f_0) \frac{d}{dx^i_j}.$$

Equation (28) immediately implies

$$\chi(d_H A) = 0, \quad \text{for any LDO } A : [(\Omega_*(J^\infty E))^{\otimes n}]_0 \to \Omega_1(J^\infty E). \tag{35}$$

Moreover, the identity $\chi(f_0 d/dx^i_j) = \chi(\chi(f_0)d/dx^i_j)$ shows that a necessary condition for the existence of a lifting is

$$\chi(f_0 \frac{d}{dx^i_j}) = 0, \ 1 \le i \le N, \ 1 \le j \le n. \tag{36}$$

Understanding equation (36) is the first step towards a complete description of the 0-cycles in $\mathrm{DEnd}_*(n)$. Basically, we can say that in the multilinear situation, $n \ge 2$, (36) implies that the terms of formal differential degree zero in the characteristic determine all the terms of higher formal differential degree. The precise statement requires some preliminaries.

First we extend the definition of the symbol to multilinear LDO, using variables ξ^i_j to represent the derivatives d/dx^i_j for $1 \le i \le N$, $1 \le j \le n$ and variables η^j_J to represent $\partial/\partial u^j_J$ for $J = (j_1, \ldots, j_N)$ and $1 \le j \le n$. Defining operators analogous to those in (20)

$$\Theta^i_j = \sum \eta^j_{J/i} \frac{\partial}{\partial \eta^j_J} \tag{37}$$

we have the commutation relations

$$\eta^j_J \xi^i_k - \xi^i_k \eta^j_J = \Theta^i_k(\eta^j_J) = \delta^j_k \Theta^i_j(\eta^j_J). \tag{38}$$

Since the symbol determines completely the LDO A, we can define a character χ', mapping symbols to symbols, with the property

$$\chi'(\sigma(A)) = \sigma(\chi(A)).$$

The commutation relation (38) implies

$$\sigma(A\frac{d}{dx^i_j}) = \sigma(A)\xi^i_j = \xi^i_j * \sigma(A) + \Theta^i_j \sigma(A),$$

where

$$\xi^i_j * \Big(p(\mathbf{x},\mathbf{u})(\xi_1)^{I_1}\cdots(\xi_n)^{I_n}(\eta^1)^{\alpha_1}\cdots(\eta^n)^{\alpha_n}\Big) := p(\mathbf{x},\mathbf{u})\xi^i_j(\xi_1)^{I_1}\cdots(\xi_n)^{I_n}(\eta^1)^{\alpha_1}\cdots(\eta^n)^{\alpha_n}.$$

Since $d/d\mathbf{y}_j = d/d\mathbf{x}_j$ for $2 \le j \le n$ we use the same symbol ξ^i_j for d/dy^i_j, but we define a new symbol ζ^i corresponding to the operator d/dy^i.

Proposition 4.4. *If A is an n-linear LDO, for $n \ge 2$, then $\chi(Ad/dx^i_j) = 0$ if and only if the symbol character $\chi'(\sigma(A))$ satisfies, for $1 \le i \le N$, the following system of equations:*

$$\sum_{j=1,\ldots,n} \Theta^i_j(\chi'(\sigma(A)) = \frac{d}{dx^i}\chi'(\sigma(A) \tag{39}$$

$$\Theta^i_j(\chi'(\sigma(A))) = -\xi^i_j * \chi'(\sigma(A)), \ \text{for } j \ge 2. \tag{40}$$

Proof. First we use the commutation relations to rewrite

$$\chi'(\sigma(A)\xi^i_j) = \chi'(\xi^i_j * \sigma(A) + \Theta^i_j(\sigma(A))) = \chi'(\xi^i_j * \sigma(A)) + \chi'(\Theta^i_j(\sigma(A))).$$

Then using $\zeta^i = \xi^i_1 + \cdots + \xi^i_n$ and the identities

$$\chi'(\zeta^i * \sigma(A)) = -\frac{d}{dx^i}\chi'(\sigma(A)) \quad \text{and} \quad \chi'(\Theta^i_j(\sigma(A))) = \Theta^i_j(\chi'(\sigma(A)),$$

we deduce

$$\begin{aligned} \chi'(\sigma(A)\zeta^i) &= \chi'(\zeta^i * \sigma(A)) + \sum_{j=1,\dots n} \Theta^i_j(\chi'(\sigma(A))) \\ &= -\frac{d}{dx^i}\chi'(\sigma(A)) + \sum_{j=1,\dots n} \Theta^i_j(\chi'(\sigma(A))), \text{ and} \\ \chi'(\sigma(A)\xi^i_j) &= \chi'(\xi^i_j * \sigma(A)) + \Theta^i_j(\chi'(\sigma(A))), \text{ for } 2 \le j \le n. \end{aligned} \tag{41}$$

If $\chi(Ad/dx^i_j) = 0$ for $1 \le j \le n$, then the left side of each of these equations is zero and rewriting the resulting equations gives the equations in the statement of the proposition. ■

The crucial fact in understanding equations (39) and (40) is that Θ^i_j is a derivation taking η^j_J to $\eta^j_{J/i}$, thus lowering formal differential degree but preserving the total homogeneity in all the variables η^j_J. There is a simple ordering on the symbol monomials $(\eta)^\alpha = \Pi_J(\eta_J)^{\alpha(J)}$ for a linear LDO, i.e. $n = 1$. It is defined by first lexicographically ordering the indices J, second, representing the exponent α by the sequence of values $\{\alpha(J)\}$ (with finitely many non-zero terms), and third, using the lexicographical ordering on these sequences, reading as in Hebrew from right to left, that is, beginning with the highest nonzero terms. For example, $\eta^n = (\eta^n_{(0,\dots,0)})$, of formal differential degree zero, is minimal among terms of homogeneity n because it corresponds to the sequence $(n, 0, 0, \dots)$ which is less than any sequence with a nonzero value beyond the first term. The above simple order induces a partial order on the symbol monomials $\Pi_{j,J}(\eta^j_J)^{\alpha_j(J)}$ with the property that $\Pi_{j,J}(\eta^j_J)^{\alpha_j(J)}$ is minimal among terms of the same homogeneity if the formal differential degree of all α_j's is zero.

Relative to this ordering, the operators Θ^i_j all have the effect of lowering the order. To simplify notation we introduce the symbols $\hat{I} = (I_1, \dots I_n)$ and $\hat{\alpha} = (\alpha_1, \dots, \alpha_n)$. Thus the coefficient $p_{\hat{I},\hat{\alpha}}(\mathbf{x}, \mathbf{u})$ appears in the expressions $\Theta^i_j\chi'(\sigma(A))$ on the left of (39) and (40) multiplied by a monomial $\xi^{\hat{I}}\eta^{\hat{\alpha}'}$ where $\hat{\alpha}'$ has lower order than $\hat{\alpha}$. On the right hand side of (39) the coefficient of $\xi^{\hat{I}}\eta^{\hat{\alpha}'}$ is $\frac{d}{dx^i}p_{\hat{I},\hat{\alpha}'}$ while on the right hand side of (40) the coefficient of $\xi^{\hat{I}}\eta^{\hat{\alpha}'}$ consists of terms involving $p_{\hat{I}',\hat{\alpha}'}$ for values $\hat{I}'$ of lower order relative to the natural lexicographical order on the indices $\hat{I}$. An elementary recursion argument shows that the solution of equations (39) and (40) is determined uniquely by the coefficients of the terms $\xi^{\hat{I}}\eta^{\hat{\alpha}}$ for minimal α, that is the monomials $\Pi(\xi_j)^{I_j}(\eta^j)^{m_j}$ with no factors η^j_J of positive formal differential degree.

For example, consider a line bundle $E \to \mathbf{R}$ and a bilinear LDO, $A : [(\Omega_*(J^\infty E))^{\otimes 2}]_0 \to \Omega_0(J^\infty E)$, with symbol

$$\sigma(A) = \sum p_{i,j,a,b}(x,u)\zeta^i \xi_2^j \eta_a^1 \eta_b^2,$$

that is, first order and of homogeneity one in the u derivatives. The symbol character is

$$\chi(\sigma(A)) = \sum (-1)^i \left(\frac{d}{dx}\right)^i p_{i,j,a,b}(x,u)\xi_2^j \eta_a^1 \eta_b^2 =: \sum \chi_{a,b}(x,u,\xi_2)\eta_a^1\eta_b^2.$$

Equation (39) becomes

$$\chi_{a+1,b} + \chi_{a,b+1} = \frac{d\chi_{a,b}}{dx}$$

and equation (40) becomes

$$\chi_{a,b+1} = -\xi_2 * \chi_{a,b}.$$

Clearly $\chi_{0,0}$ determines all the $\chi_{a,b}$ for $a, b \geq 0$.

Let us go back to the discussion of the lifting problem. In Figure 1 we present a diagram describing our situation. It contains maps $\mathcal{B}_1, \mathcal{B}_0, \mathcal{B}_{-1}$, a_1 and a_0 which we now define.

For $f = \{f_s\} \in \mathrm{DEnd}_0(n)$ (the notation of (32)) define a projection on the lowest term $\mathcal{B}_0 : \mathrm{DEnd}_0(n) \to \mathrm{LDO}(n)$ by $\mathcal{B}_0(f) := f_0$. For $h = \{h_s\} \in \mathrm{DEnd}_1(n)$, put $\mathcal{B}_1(h) := d_H h_0 \in \mathrm{LDO}(n)$. The following lemma is an immediate consequence of (35).

Lemma 4.5. *In the situation above, $\chi(d_H h_0) = 0$, thus $\mathcal{B}_1$ can be interpreted as a map $\mathcal{B}_1 : \mathrm{DEnd}_1(n) \to \{A \in \mathrm{LDO}; \chi(A) = 0\}$.*

The map $\mathcal{B}_{-1} : \mathrm{DEnd}_{-1}(n) \to \bigoplus_{\substack{1\leq i\leq N\\ 1\leq j\leq n}} \mathrm{LDO}(n)_{i,j} \cong \mathrm{LDO}(n)^{\oplus Nn}$ is defined as follows. Let $g = \{g_s\} \in \mathrm{DEnd}_{-1}(n)$, then

$$g_1 : [(\Omega_*(J^\infty E))^{\otimes n}]_1 \to \Omega_0(J^\infty E).$$

Using the description (34) of $[(\Omega_*(J^\infty E))^{\otimes n}]_1$ as a direct sum of pieces $\mathrm{Loc}(E)^{\otimes n}_{i,j}$, the (i,j)-th component of $\mathcal{B}_{-1}(g)$ is defined to be the characteristic of the restriction $g_1|_{\mathrm{Loc}(E)^{\otimes n}_{i,j}}$.

Let $a_1 : \{A \in \mathrm{LDO}(n);\ \chi(A) = 0\} \hookrightarrow \mathrm{LDO}(n)$ be the inclusion and, finally, the map $a_0 : \mathrm{LDO}(n) \to \mathrm{LDO}(n)^{\oplus Nn}$ is given by $a_0(A)_{i,j} := \chi(A \circ d/dx^i_j)$.

Lemma 4.6. *The sequence $\{A \in \mathrm{LDO}(n);\ \chi(A) = 0\} \xrightarrow{a_1} \mathrm{LDO}(n) \xrightarrow{a_0} \mathrm{LDO}^{\oplus Nn}$ is a differential chain complex.*

Proof. We must show that $a_0 a_1 = 0$, which is the same as to prove that $\chi(A \circ \frac{d}{dx^i_j}) = 0$ whenever $\chi(A) = 0$. This follows immediately from (41). ∎

Lemma 4.7. *All horizontal maps in The un***End***ing ladder (Figure 1) are maps of chain complexes.*

$$
\begin{tikzcd}
\vdots \arrow[d, "\delta"] & \vdots \arrow[d] \\
\mathrm{DEnd}_2(n) \arrow[r] \arrow[d, "\delta"] \arrow[dr, phantom, "\boxed{1}"] & 0 \arrow[d] \\
\mathrm{DEnd}_1(n) \arrow[r, "\mathcal{B}_1"] \arrow[d, "\delta"] \arrow[dr, phantom, "\boxed{2}"] & \{A \in \mathrm{LDO}(n);\ \chi(A) = 0\} \arrow[d, "a_1"] \\
\mathrm{DEnd}_0(n) \arrow[r, "\mathcal{B}_0"] \arrow[d, "\delta"] \arrow[dr, phantom, "\boxed{3}"] & \mathrm{LDO}(n) \arrow[d, "a_0"] \\
\mathrm{DEnd}_{-1}(n) \arrow[r, "\mathcal{B}_{-1}"] \arrow[d, "\delta"] & \mathrm{LDO}(n)^{\oplus Nn} \arrow[d] \\
\mathrm{DEnd}_{-2}(n) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
$$

Figure 1: Jacob's un**End**ing ladder.

Proof. We must show that the diagrams $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ commute.

$\boxed{1}$ *commutes.* This means proving that $\mathcal{B}_1\delta(l) = 0$, for each $l \in \mathrm{DEnd}_2(n)$. But $(\delta l)_0 = (d_H l)_0$, thus $\mathcal{B}_1\delta(l) = 0$ follows from $d_H^2 = 0$.

$\boxed{2}$ *commutes.* The equality $a_1\mathcal{B}_1(h) = \mathcal{B}_0\delta(h)$ follows immediately from definitions.

$\boxed{3}$ *commutes.* If $f = \{f_s\} \in \mathrm{DEnd}_0(n)$ then, by definition, $[a_0\mathcal{B}_0(f)]_{i,j} = \chi(f_0 \circ d/dx^i_j)$. On the other hand, $\mathcal{B}_{-1}\delta(f)$ is computed as the characteristic of the restriction of $f_0 d_H + d_H f_1$ to $\mathrm{Loc}(E)_{i,j}$. The second term gives, by Lemma 4.5, zero, while the first term gives $\chi(f_0 \circ d/dx^i_j)$, as it should. ■

Theorem 4.8. *In the diagram of Figure 1,*

(i) the complex $(\mathrm{DEnd}_*(n), \delta)$ *is acyclic in positive dimensions,* $H_{>0}(\mathrm{DEnd}_*(n), \delta) = 0$*,*

(ii) the map $\mathcal{B}_*$ *induces an isomorphism of the* 0*th homology group,*

$$H_0(\mathrm{DEnd}(n)) \cong \frac{\{A \in \mathrm{LDO}(n);\ \chi(Ad/dx^i_j) = 0,\ \textit{for all } 1 \le i \le N, 1 \le j \le n\}}{\{A \in \mathrm{LDO}(n);\ \chi(A) = 0\}}.$$

Moreover, the map $\mathcal{B}_0$ *is an epimorphism of cycles,* $\mathcal{B}_0(Z_0(\mathrm{DEnd}(n))) = \mathrm{Ker}(a_0)$*.*

Proof. See the next section.

Corollary 4.9. *A LDO* $A : [\Omega_*(J^\infty E)^{\otimes n}]_0 \to \Omega_0(J^\infty E)$ *can be lifted to a sequence* $f_s : [\Omega_*(J^\infty E)^{\otimes n}]_s \to \Omega_s(J^\infty E)$ *with* $f_0 = A$ *if and only if*

$$\chi(A\frac{d}{dx^i_j}) = 0, \tag{42}$$

for each $1 \le i \le N$, $1 \le j \le n$*. In the case* $\chi(A) = 0$*, the lift can be chosen in such a way that*

$$f_s = 0,\ \textit{for } s \ge 2. \tag{43}$$

Proof. The first part of the corollary claims the existence of an $f \in \mathrm{DEnd}_0(n)$, $\delta f = 0$, with $\mathcal{B}_0(f) = A$. But (42) means that $a_0(A) = 0$ and the existence of f follows from the fact that $\mathcal{B}_0$ is an epimorphism of cycles.

Let us prove the second part of the corollary. Suppose that $\chi(A) = 0$ and let $\overline{f} = \{\overline{f}_s\}$ be a lift of A. Since $A \in \mathrm{Im}(a_1)$, there exists $h \in \mathrm{DEnd}_1(n)$ such that $\overline{f} = \delta h$. This means that $A = \overline{f}_0 = d_H h_0$. One immediately sees that $f = \{f_s\}$ with $f_0 = A$, $f_1 := h_0 d_H^{\otimes n}$ and $f_s = 0$ for $s \ge 2$ is a lift of A. ■

5. Proofs.

This section is devoted to the proof of Theorem 4.8. The basic tool will be the following de Rham complex with operator coefficients.

Definition 5.1. *The* operator complex $O^*(n) = (O^*(n), d)$ *is the complex of de Rham forms on* $J^\infty E$ *with coefficients in* $\mathrm{LDO}(n)$. *The differential* d *is given by*

$$d(A(d\mathbf{x})^\epsilon) = \sum_{1 \le i \le N} \left(\frac{d}{dx^i} A \right) dx^i \wedge (d\mathbf{x})^\epsilon. \tag{44}$$

Let $J : O^N(n) \to \mathrm{LDO}(n)$ be the map $J(A dx^1 \wedge \cdots \wedge dx^N) := A$.

Theorem 5.2. *The complex* $(O^*(n), d)$ *is acyclic in degrees* $< N$, *that is* $H^{<N}(O^*(n), d) = 0$, *while the map* J *induces an isomorphism*

$$H^N(O^*(n), d) \cong \frac{\mathrm{LDO}(n)}{\{A \in \mathrm{LDO}(n);\ \chi(A) = 0\}}.$$

Notice the following rather surprising fact: the operator complex $(O^*(n), d)$ is acyclic in degree 0, though the 'ordinary' horizontal de Rham complex $(\Omega^{*,0}(J^\infty E), d_H)$ is not, $H^0(\Omega^{*,0}(J^\infty E), d_H) = \mathbf{R}$! This follows from (and implies) the following stunning property of local differential operators:

$$\text{if } A \in \mathrm{LDO}(n) \text{ and } d/dx^i \circ A = 0,\ 1 \le i \le N, \text{ then } A = 0.$$

This will not be true if we remove the 'convergence property' (25). As an example, take the operator

$$A := 1 - x\frac{d}{dx} + \frac{1}{2}x^2\frac{d^2}{dx^2} - \frac{1}{6}x^3\frac{d^3}{dx^3} + \cdots$$

in one space and no vertical variables. It clearly satisfies $d/dx \circ A = 0$. Observe that, for a polynomial f, $A(f) = f(0)$.

Proof of Theorem 5.2. Let us look more closely at the structure of the differential in the complex $(O^*(n), d)$. If $A(d\mathbf{x})^\epsilon \in O(n)$, where $A \in \mathrm{LDO}(n)$ is as in (27), then

$$d(A(d\mathbf{x})^\epsilon) = (d_1 + d_2)(A(d\mathbf{x})^\epsilon)$$

with

$$d_1(A(d\mathbf{x})^\epsilon) = \sum \frac{dq_{I_1, I_2, \ldots, I_n; \alpha_1, \ldots, \alpha_n}}{dx^i} \left(\frac{d}{d\mathbf{y}_1}\right)^{I} \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n} dx^i \wedge (d\mathbf{x})^\epsilon$$

and

$$d_2(A(d\mathbf{x})^\epsilon) = \sum q_{I_1, I_2, \ldots, I_n; \alpha_1, \ldots, \alpha_n} \left(\frac{d}{d\mathbf{y}_1}\right)^{iI} \left(\frac{d}{d\mathbf{y}_2}\right)^{I_2} \cdots \left(\frac{d}{d\mathbf{y}_n}\right)^{I_n} \left(\frac{\partial}{\partial \mathbf{u}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \mathbf{u}_n}\right)^{\alpha_n} dx^i \wedge (d\mathbf{x})^\epsilon.$$

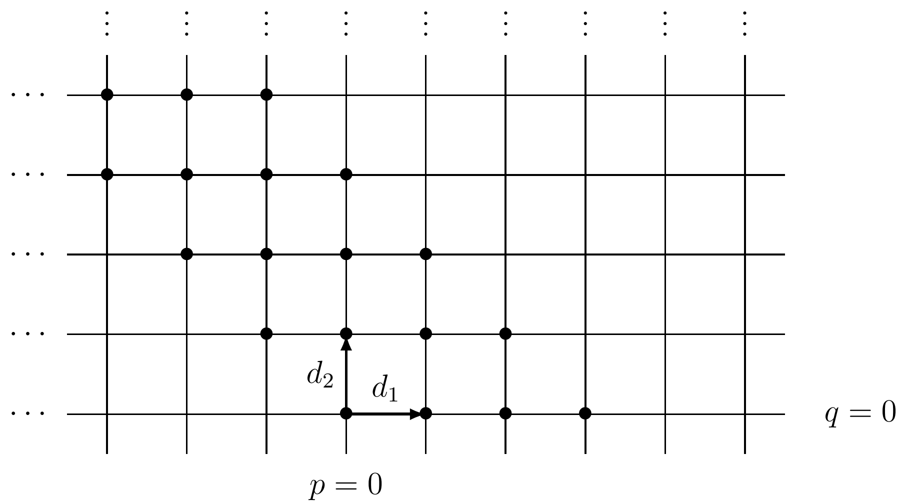

Figure 2: The shape of the bicomplex $\mathbf{E} = (E^{*,*}, d = d_1 + d_2)$ for $N = 3$. Solid dots indicate nontrivial entries.

Let us decompose $O^*(n) = \bigoplus_{p+q=*} O^{p,q}(n)$, where $O^{p,q}$ consists of $A(d\mathbf{x})^\epsilon \in O^{p+q}(n)$ such that $A \in \mathrm{LDO}(n)$ in the polarized form of (27) contains exactly q instances of total derivatives d/dy. Denote for simplicity $E^{p,q} := O^{p,q}(n)$. Then

$$d_1 : E^{p,q} \to E^{p+1,q} \text{ and } d_2 : E^{p,q} \to E^{p,q+1}.$$

We are going to study the properties of the bicomplex $\mathbf{E} := (E^{*,*}, d = d_1 + d_2)$. Observe that $\mathbf{E}$ is not contained in the first quadrant, but

$$E^{p,q} \neq 0 \text{ for } 0 \leq p + q \leq N,\ q \geq 0.$$

The shape of the bicomplex is indicated in Figure 2. For any fixed $q, m \geq 0$, let V_q^m be, , the $\mathbf{R}$-vector space with the basis

$$\{e^{I,I_2,\ldots,I_n;\alpha_1,\ldots,\alpha_n};\ |I| = q \text{ and } \sum_{1 \leq i \leq n} \deg_f(\alpha_i) \leq m\}$$

and let $V_q := \varprojlim V_q^m$, the inverse limit of the system of natural projections $V_q^m \to V_q^n$, $m \geq n$. Then the horizontal complex $(E^{*,q}, d_1)$ is the tensor product of V_q and the horizontal de Rham complex $(\Omega^{*,0}(J^\infty E), d_H)$. It follows from the description of the cohomology of this complex [1] that

$$H^s(E^{*,q}, d_1) = \begin{cases} V_q, & \text{for } s = -q, \\ 0, & \text{for } -q < s < N - q, \text{ and} \\ V_q \otimes H^N(\Omega^{*,0}(J^\infty E), d_H), & s = N - q. \end{cases} \tag{45}$$

One is tempted to calculate the homology of the bicomplex $\mathbf{E}$ using an obvious spectral sequence, but, since the bicomplex is not only in the first quadrant, one must be very careful

about the convergence. The calculation just given of the first stage of the spectral sequence of a double complex filtered by rows, the complex with operator d_1, gives an E_1 term with nonzero entries along both counterdiagonals $p+q=0$ and $p+q=N$. It would seem from this that the cohomology of the double complex is far from trivial in dimension zero. The reason that this is not the case is that the bicomplex is not a first quadrant one and using the filtration by columns would lead us to constructing a cocycle which would be an infinite sum with increasing q, contradicting the finiteness condition (25).

On the other hand, consider the spectral sequence for the filtration by columns. Let W^m be, for $m \geq 0$, the $\mathrm{Loc}(E)$-vector space with the basis

$$\{f^{I_2,\dots,I_n;\alpha_1,\dots,\alpha_n};\ \sum_{1\leq i\leq n} \deg_f(\alpha_i) \leq m\}$$

and let $W := \varinjlim W^m$, of course, $W \cong \mathrm{LDO}^0(n)$. Then the p-th column of the first stage of this spectral sequence, with differential given by d_2, is isomorphic to W tensored with the Koszul complex

$$K_p := \cdots \longrightarrow S^q(\zeta)\otimes \wedge^{p+q}(d\mathbf{x}) \xrightarrow{d_K} S^{q+1}(\zeta)\otimes \wedge^{p+q+1}(d\mathbf{x}) \longrightarrow \cdots$$

$$d_K(f\otimes(d\mathbf{x})^\epsilon) = \sum_{i=1,\dots,N} \zeta^i f\otimes dx^i \wedge (d\mathbf{x})^\epsilon$$

where $0 \leq q \leq N-p$, $S^q(\zeta)$ is the $\mathbf{R}$-vector space of homogeneous polynomials of degree q in the variables $\zeta^1,\dots,\zeta^N$, and $\bigwedge^{p+q}(d\mathbf{x})$ is the degree $p+q$ component of the $\mathbf{R}$-exterior algebra on $dx^1,\dots,dx^N$. It is easy to see, for example using the contracting homotopy which is defined on terms $f\otimes(d\mathbf{x})^\epsilon \in S^q(\zeta)\otimes \wedge^{p+q}(d\mathbf{x})$ by

$$f\otimes(d\mathbf{x})^\epsilon \longmapsto \frac{1}{N-p}\sum_{1\leq i\leq N}\frac{\partial f}{\partial \zeta^i}\otimes\iota\left(\frac{\partial}{\partial x^i}\right)(d\mathbf{x})^\epsilon,$$

that the cohomology of the Koszul complex is trivial for $p<N$ and one dimensional with basis $dx^1\wedge\cdots\wedge dx^N$ for $p=N$. Thus

$$\begin{aligned} H^q(E^{p,*},d_2) &= 0,\ \text{for } 0<q, \text{ or } 0=q \text{ and } p<N, \text{ and}\\ H^0(E^{N,*},d_2) &= W \cong \mathrm{LDO}^0(n). \end{aligned}$$

In this case, since E_1 has only one nonzero term, we can conclude that $H^N(O^*(n),d) \cong \mathrm{LDO}^0(n)$, but we shall give a direct proof anyway.

Suppose that $c = c_{k,0} + c_{k-1,1} + \cdots c_{k-u,u}$ is a degree k cycle, $0<k\leq N$, $c_{k-i,i}\in E^{k-i,i}$, $i\geq 0$. By a degree argument, $d_2(c_{k-u,u}) = 0$. Let $u\geq 1$. By the acyclicity of $(E^{p,*},d_2)$, there exists an $\alpha\in E^{k-u,u-1}$ such that $d_2(\alpha) = c_{k-u,u}$. We then replace c by $c-(d_1+d_2)(\alpha)$, which is in the same homology class, but which has no component in $E^{k-u,u}$. Repeating this process as many times as necessary we conclude that we could in fact assume that $u=0$, or $c\in E^{k,0}$. This means, since we assumed c to be a cycle, that $d_1 c = 0$ and $d_2 c = 0$. Observe

that this reduction works for $k = 0$ as well; we may immediately conclude that $c_{-u,u} = 0$ since d_2 is a monomorphism on $E^{-u,u}$.

Now, if $k < N$, we immediately conclude that $c = 0$, because d_2 is a monomorphism on $E^{k,0}$. This proves the acyclicity of $O^*(n)$ in degrees $< N$.

If $k = N$, then $c = Adx^1 \wedge \cdots \wedge dx^N$, where $A \in \mathrm{LDO}(n)$ contains no d/dx^i; we denoted the set of all such LDO's by $\mathrm{LDO}^0(n)$. Such c cannot be a nontrivial boundary. To see this, let $c = (d_1 + d_2)b$, with $b = b_{k-1,0} + b_{k-2,1} + \cdots + b_{k-v-1,v}$, $b_{k-i-1,i} \in E^{k-i-1,i}$. The 'leading' term $b_{k-v-1,v}$ must be a d_2-cycle hence a d_2-boundary and we may go 'down the staircase' as above and assume $b = b_{k-1,0}$. Then $d_1(b) = c$ and $d_2(b) = 0$, which implies $b = 0$, since d_2 is a monomorphism on $E^{k-1,0}$. We proved

$$H^N(O^*(n), d) \cong \mathrm{LDO}^0(n),$$

which, together with (29), finishes the proof. ■

Proof of Theorem 4.8. For simplicity, we will explicitly specify the range of summations only when it will not be obvious. Suppose $f \in \mathrm{DEnd}_k(n)$. This means that f is a sequence of maps

$$f_s : [(\Omega^*(J^\infty E))^{\otimes n}]_s \to \Omega^{s+k}(J^\infty E).$$

The space $[(\Omega^*(J^\infty E))^{\otimes n}]^s$ is spanned by elements of the form

$$(h_1(d\mathbf{x}_1)^{\epsilon_1}) \otimes \cdots \otimes (h_n(d\mathbf{x}_n)^{\epsilon_n}),\ h_j \in C^\infty(\mathbf{R}^N),\ 1 \le j \le n,$$

where the subscript indicates to which copy of $\mathbf{R}^N$ the corresponding object applies, and $\sum_{1\le j\le n} |\epsilon_j| = nN - s$ (remember the regrading (30)). With this notation, f_s acts by

$$f_s((h_1(d\mathbf{x}_1)^{\epsilon_1}) \otimes \cdots \otimes (h_n(d\mathbf{x}_n)^{\epsilon_n})) = \sum_\epsilon A^{\epsilon_1,\ldots,\epsilon_n}_{s,\epsilon}(h_1, \ldots, h_n)(d\mathbf{x})^\epsilon,$$

with some $A^{\epsilon_1,\ldots,\epsilon_n}_{s,\epsilon} \in \mathrm{LDO}(n)$. In other words, $f = \{f_s\}$ is represented by the system

$$\{F^{\epsilon_1,\ldots,\epsilon_n}_s \in O^{N-s-k}(n)\},\ F^{\epsilon_1,\ldots,\epsilon_n}_s := \sum_\epsilon A^{\epsilon_1,\ldots,\epsilon_n}_{s,\epsilon}(d\mathbf{x})^\epsilon, \tag{46}$$

where $\sum_j |\epsilon_j| = nN - s$ and $\max(-k, 0) \le s \le \min(nN, N-k)$. The last inequality simplifies, for $k \ge 0$, to $0 \le s \le N - k$.

Let us try to understand how the differential δ in $\mathrm{DEnd}_*(n)$, defined by (33), works. We have

$$(\delta f)_s((h_1(d\mathbf{x}_1)^{\epsilon_1}) \otimes \cdots \otimes (h_n(d\mathbf{x}_n)^{\epsilon_n})) = \sum_{i,\epsilon} \frac{d}{dx^i} A^{\epsilon_1,\ldots,\epsilon_n}_{s,\epsilon}(h_1, \ldots, h_n)dx^i \wedge (d\mathbf{x})^\epsilon$$
$$- \sum_{i,j,\delta} (-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)} \cdot A^{\epsilon_1,\ldots,i\epsilon_j,\ldots,\epsilon_n}_{s,\delta}(h_1, \ldots, \frac{d}{dx^i} h_j, \ldots, h_n)(d\mathbf{x})^\delta.$$

In the second term, $i\epsilon_j$ has the obvious meaning similar to that of iJ, see (13). The above formula can be written in terms of the expressions (46) as

$$\delta\{F_s^{\epsilon_1,\dots,\epsilon_n}\} = \{dF_s^{\epsilon_1,\dots,\epsilon_n} - \sum_{i,j}(-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)}\cdot F_{s-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i}\} \tag{47}$$

where the differential d in the first term of the right hand side is the differential of the operator complex (44). Let us prove that $\mathrm{DEnd}_*(n)$ *is acyclic in positive dimensions.* If $\{F_s^{\epsilon_1,\dots,\epsilon_n}\} \in \mathrm{DEnd}_k(n)$ is a cycle, $k > 0$, then, by (47),

$$0 = dF_s^{\epsilon_1,\dots,\epsilon_n} - \sum_{i,j}(-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)}\cdot F_{s-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i}, \tag{48}$$

for all s, $\epsilon_1,\dots,\epsilon_n$. We are looking for $\{H_s^{\epsilon_1,\dots,\epsilon_n}\} \in \mathrm{DEnd}_{k+1}(n)$ such that

$$F_s^{\epsilon_1,\dots,\epsilon_n} = dH_s^{\epsilon_1,\dots,\epsilon_n} + \sum_{i,j}(-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)}\cdot H_{s-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i}. \tag{49}$$

Let us solve this equation inductively. For $s = 0$ it reduces to

$$F_0^{\epsilon_1,\dots,\epsilon_n} = dH_0^{\epsilon_1,\dots,\epsilon_n}, \tag{50}$$

which must be solved in $O^{N-k}(n)$. Equation (48) with $s = 0$ says that $dF_0^{\epsilon_1,\dots,\epsilon_n} = 0$ and the existence of $H_0^{\epsilon_1,\dots,\epsilon_n}$ follows from the acyclicity of the operator complex (Theorem 5.2).

Suppose we have solved (49) for all $s < r$ and try to solve it for $s = r$. By the acyclicity of the operator complex in dimension $N - r - k$, it is enough to verify that

$$F_r^{\epsilon_1,\dots,\epsilon_n} - \sum_{i,j}(-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)}\cdot H_{r-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i} \tag{51}$$

is closed in $O^{N-r-k}(n)$. By (48), $dF_r^{\epsilon_1,\dots,\epsilon_n}$ equals

$$\sum_{i,j}(-1)^{(k+|\epsilon_1|,\dots,|\epsilon_{j-1}|)}\cdot F_{r-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i},$$

while the differential of the second term of (51) equals, by the inductive assumption, to

$$-\sum_{i,j}(-1)^{(k+|\epsilon_1|+\cdots+|\epsilon_{j-1}|)}\cdot F_{r-1}^{\epsilon_1,\dots,i\epsilon_j,\dots,\epsilon_n} \circ \frac{d}{dx_j^i} + \sum_{\substack{i,j,k,l\\ j\neq l}}(-1)^{\sigma(j,l)}\cdot H_{r-2}^{\epsilon_1,\dots,i\epsilon_j,\dots,k\epsilon_l,\dots,\epsilon_n} \circ \frac{d}{dx_j^i}\frac{d}{dx_l^k} \tag{52}$$

where

$$\sigma(j,l) := \begin{cases} 1 + |\epsilon_j| + \cdots + |\epsilon_{l-1}|, & \text{for } j < l, \text{ and} \\ |\epsilon_l| + \cdots + |\epsilon_{j-1}|, & \text{for } j > l. \end{cases}$$

It is immediate to conclude that the second term of (52) is zero. Summing up the above informations we see that the form in (51) is indeed closed, and the induction may go on.

Let us assume $f = \{F_s^{\epsilon_1,\ldots,\epsilon_n}\} \in \mathrm{DEnd}_0(n)$. This means that $F_s^{\epsilon_1,\ldots,\epsilon_n} \in O^{N-s}(n)$, $0 \leq s \leq N$, and $\sum_j |\epsilon_j| = nN - s$. For $s = 0$ this may happen only if $\epsilon_j = (1, \ldots, 1)$ for all $1 \leq j \leq N$, and the system $\{F_0^{\epsilon_1,\ldots,\epsilon_n}\}$ boils down to one element $F_0 \in O^N(n)$. Let us write $F_0 = f_0 dx^1 \wedge \ldots \wedge dx^N$ with some $f_0 \in \mathrm{LDO}(n)$. By definition, $\mathcal{B}_0(f) = f_0$. As before, try to solve (49) inductively. For $s = 0$ it reduces to

$$F_0 = dH_0,\ H_0 \in O^{N-1}(n),$$

which can be, by Theorem 5.2, solved if and only if $\chi(f_0) = 0$, which is the same as $\mathcal{B}_0(f) \in \mathrm{Im}(a_1)$. If this is the case, the induction may go on, by the acyclicity of the operator complex. We proved that $H_0(\mathcal{B}_*)$ *is a monomorphism.*

Let us prove that $\mathcal{B}_0(Z_0(\mathrm{LDO}(n))) = \mathrm{Ker}(a_0)$. Suppose $f_0 \in \mathrm{LDO}(n)$ with $a_0(f_0) = 0$. We are looking for a cycle $f = \{F_s^{\epsilon_1,\ldots,\epsilon_n}\} \in \mathrm{DEnd}_0(n)$ such that $F_0 = f_0 dx^1 \wedge \ldots \wedge dx^N$. We construct such a cycle by inductively solving (48):

$$dF_s^{\epsilon_1,\ldots,\epsilon_n} = \sum_{i,j} (-1)^{(|\epsilon_1|+\cdots+|\epsilon_{j-1}|)} \cdot F_{s-1}^{\epsilon_1,\ldots,i\epsilon_j,\ldots,\epsilon_n} \circ \frac{d}{dx^i_j}, \tag{53}$$

in $O^{N-s+1}(n)$, for $s \geq 1$. We already observed that $F_0^{\epsilon_1,\ldots,\epsilon_n}$ reduces to one element, F_0. Thus (53) can be, for $s = 1$, written as

$$dH^{i,j} = (-1)^{N(j-1)} \cdot F_0 \circ \frac{d}{dx^i_j},\ 1 \leq i \leq N,\ 1 \leq j \leq n, \tag{54}$$

where $H^{ij} := F_1^{\epsilon_1,\ldots,\epsilon_n}$ with

$$\epsilon_k := \begin{cases} (1, \ldots, 1), & \text{for } k \neq j, \text{ and} \\ (1, \ldots, 1, 0, 1, \ldots, 1)\ /0 \text{ at the } i\text{-th position}/, & \text{for } k = j. \end{cases}$$

By Theorem 5.2, equation (54) can be solved if and only if $\chi(F_0 \circ d/dx^i_j) = 0$, $1 \leq i \leq N$, $1 \leq j \leq n$, which is the same as to say that $f_0 \in Ker(a_0)$. The inductive construction then may go on by the acyclicity of the operator complex. ■

6. Applications.

This paper originated in our attempts to understand the paper [2] with the problem of defining a lifting of a Poisson structure on the algebra of functionals to a strong homotopy Lie structure on the horizontal complex. We review the situation discussed in that paper. First let us recall a basic definition and lemma from [2].

Definition 6.1. *A* local functional

$$\mathcal{L}[\phi] = \int L(\mathbf{x}, \phi(\mathbf{x}))dx^1 \wedge \cdots \wedge dx^N = \int (j^\infty \phi)^* L(\mathbf{x}, \mathbf{u})dx^1 \wedge \cdots \wedge dx^N \tag{55}$$

is the integral of the pull-back of an element of $L(\mathbf{x}, \mathbf{u})dx^1 \wedge \cdots \wedge dx^N \in \Omega^{N,0}(J^\infty E)$ by the section $j^\infty \phi$ of $J^\infty E$ corresponding to the section ϕ of E, where we assume that $L(\mathbf{x}, 0) = 0$. The integral will always be well-defined since $(j^\infty \phi)^* L(\mathbf{x}, \mathbf{u})dx^1 \wedge \cdots \wedge dx^N$ is a compactly supported smooth N-form on $\mathbf{R}^N$ for ϕ a section with compact support, since the coefficient vanishes outside the support of ϕ.

The space of local functionals $\mathcal{F}$ *is the vector space of equivalence classes of local functionals, where two local functionals are equivalent if they agree for all sections of compact support.*

The correspondence between the functional and the corresponding element of $\Omega^{N,0}(J^\infty E)$ is not one to one, as we see from the following lemma.

Lemma 6.2. *The vector space of local functionals $\mathcal{F}$ is isomorphic to the cohomology group*

$$H^N(\Omega^{*,0}(J^\infty E), d_H) = \Omega^{N,0}(J^\infty E)/d_H(\Omega^{N-1,0}(J^\infty E)).$$

The isomorphism of the lemma is induced by the correspondence

$$\Omega^{N,0}(J^\infty E) \ni f(\mathbf{x}, \mathbf{u})dx^1 \wedge \cdots \wedge dx^N \longleftrightarrow \mathcal{L} \in \mathcal{F},$$

where $\mathcal{L}$ is the functional corresponding to the form $L(\mathbf{x}, \mathbf{u})dx^1 \wedge \cdots \wedge dx^N$, with $L(\mathbf{x}, \mathbf{u}) := f(\mathbf{x}, \mathbf{u}) - f(\mathbf{x}, 0)$.

We cite from the introduction to [2]:

"The approach to Poisson brackets in this context, pioneered by Gel'fand, Dickey and Dorfman (see [9, 8, 10, 11, 12, 15, 6] for reviews), is to consider the Poisson brackets for local functionals as being induced by brackets for local functions, which are not necessarily strictly Poisson. We will analyze here in detail the structure of the brackets for local functions corresponding to the Poisson brackets for local functionals. More precisely, we will show that these brackets will imply higher order brackets combining into a strong homotopy Lie algebra."

A suitable bracket for local functions, which is not necessarily strictly Poisson, is given by $\tilde{l}_2 \in O^N(2)$ considered as a map from $\Omega^{N,0}(J^\infty E) \otimes \Omega^{N,0}(J^\infty E)$ to $\Omega^{N,0}(J^\infty E)$ such that:

(i) $\tilde{l}_2(\alpha, d_H\beta) \in d_H\Omega^{N-1,0}(J^\infty E)$ for all $\alpha \in \Omega^{N,0}(J^\infty E)$ and $\beta \in \Omega^{N-1,0}(J^\infty E)$,

(ii) $\tilde{l}_2(\alpha, \beta) + \tilde{l}_2(\beta, \alpha) \in d_H(\Omega^{N-1,0}(J^\infty E))$, and

(iii)

$$\sum_{\sigma \in unsh(2,1)} (\tilde{l}_2 \circ_\sigma \tilde{l}_2)(\alpha_1, \alpha_2, \alpha_3) :=$$

$$:= \sum_{\sigma \in unsh(2,1)} e(\sigma)\epsilon(\sigma)\tilde{l}_2(\tilde{l}_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)}) \in d_H\Omega^{N-1,0}(J^\infty E)$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Omega^{N,0}(J^\infty E)$, where *unsh*$(k,p)$ is the set of permutations σ satisfying

$$\underbrace{\sigma(1) < \ldots < \sigma(k)}_{\text{first } \sigma \text{ hand}} \quad \text{and} \quad \underbrace{\sigma(k+1) < \ldots < \sigma(k+p),}_{\text{second } \sigma \text{ hand}}$$

$e(\sigma)$ is the Koszul sign and $\epsilon(\sigma)$ is the standard sign of a permutation. The meaning of the last condition is that $\tilde{l}_2$ satisfies the Jacobi identity up to a boundary, that is

$$\tilde{l}_2(\tilde{l}_2(\alpha_1, \alpha_2), \alpha_3) + \tilde{l}_2(\tilde{l}_2(\alpha_2, \alpha_3), \alpha_1) + \tilde{l}_2(\tilde{l}_2(\alpha_3, \alpha_1), \alpha_2) \in d_H \Omega^{N-1,0}(J^\infty E).$$

The paper [2] proves that when the original map $\tilde{l}_2$ satisfies conditions (i), (ii) and (iii), there is a *strong homotopy Lie structure* (see [13] for relevant definitions) on the graded vector space $X_* = \Omega_*(J^\infty E)$, that is, a collection of linear, skew symmetric maps $l_k : (X^{\otimes k})_* \longrightarrow X_{*+k-2}$, $k \geq 1$, that satisfy, for any $n \geq 1$, the relation

$$\sum_{\substack{i+j=n+1 \\ \sigma \in unsh(i,n-i)}} e(\sigma)\epsilon(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0.$$

In this formulation the element l_1 of degree -1 is the boundary operator of the complex. In the case under consideration, $l_1 = d_H$.

The authors use the exactness of the complex $\Omega_*(J^\infty E)$ to define the lifting l_2 as well as to define the higher brackets, l_k. From our point of view there is a problem with their construction in that the recursive definition of l_k is based on a choice for each k-tuple of local functions, and there is no control of the class of operator being defined. For example, they argue:

"In degree zero, $\sum_{\sigma \in unsh(2,1)} e(\sigma)\epsilon(\sigma)\tilde{l}_2(\tilde{l}_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})$ is equal to a boundary b in X_0 by condition (iii). There exists an element $z \in X_1$ with $l_1 z = b$ and so we define $\tilde{l}_3(x_1, x_2, x_3) = -z$."

This "pointwise" construction is not sufficient to guarantee if $\tilde{l}_2$ is a LDO that the lift $\tilde{l}_3$ will be also. Given a higher bracket l_k which has been partially defined up to degree j

$$l_k : \bigoplus_{i \leq j} (X^{\otimes k})_i \longrightarrow \bigoplus_{i \leq j} X_{i+k-2},$$

similar arguments are invoked to extend it to

$$l_k : (X^{\otimes k})_{j+1} \longrightarrow X_{j+k-1}.$$

There is no guarantee that the map so defined will satisfy $l_k \in \mathrm{DEnd}_{k-2}(k)$.

The results of Sections 4 and 5 together with Proposition 6.4 given below allow us to prove that in fact all $l_k \in \mathrm{DEnd}_{k-2}(k)$. To be more precise,

– Theorem 4.8(ii) will establish the existence of a suitable extension of $\tilde{l}_2$,

– Theorem 5.2 will allow us to define $\tilde{l}_3 \in O^{N-1}(3)$ as well as the extension to $l_3 \in \mathrm{DEnd}_1(3)$.
– Once we have defined l_3 the existence of the higher l_k follows from Theorem 4.8 (i), i.e., the acyclicity in dimensions greater than zero of $\mathrm{DEnd}_*(n)$.

Recall that the *Euler operator* $E : \Omega^{N,0}(J^\infty E) \to \mathrm{Loc}(E)$ is defined by

$$E(L(\mathbf{x},\mathbf{u})dx^1 \wedge \ldots \wedge dx^N) := \sum_I (-1)^I \left(\frac{d}{d\mathbf{x}}\right)^I \left(\frac{\partial L}{\partial u_I}\right).$$

The next well-known lemma and the subsequent proposition play an essential rôle here by allowing us to express in terms of the character of a LDO conditions such as (i), (ii) and (iii) which involve the image of d_H, that is, d_H of some unspecified forms, for a proof see [15].

Lemma 6.3. *An element $\alpha \in \Omega^{N,0}(J^\infty E)$ has the form $d_H\beta$ for $\beta \in \Omega^{N-1,0}(J^\infty E)$ if and only if $E(\alpha) = 0$.*

Proposition 6.4. *Given $A \in O^N(n)$, $A(f_1,\ldots,f_n) \in d_H(\Omega^{N-1,0}(J^\infty E))$ for all n-tuples $(f_1,\ldots,f_n) \in Loc(E)^{\otimes n}$ if and only if $\chi(A) = 0$.*

Proof. From (31) we know that there exists $B \in O^{N-1}(n)$ such that $A = d_H \circ B + \chi(A)$. Since $E \circ d_H = 0$ we have

$$E \circ \chi(A) = E \circ A. \tag{56}$$

By this equation, $\chi(A) = 0$ implies $E \circ A = 0$ and $A(f_1,\ldots,f_n) \in d_H(\Omega^{N-1,0}(J^\infty E))$ by Lemma 6.3.

To prove the opposite implication, it is enough, again by (56), to show that $E \circ \chi(A) = 0$ implies $\chi(A) = 0$. Since $\chi(A) \in \mathrm{LDO}^0(n)$, this will follow from the following lemma.

Lemma 6.5. *For $A \in \mathrm{LDO}^0(n)$, $E \circ A = 0$ if and only if $A = 0$.*

Proof. Let us discuss the linear case, $n = 1$, first. Each exponent α can be written as $\alpha = (n,\beta)$, where $n := \alpha(0,\ldots,0)$ and β is the rest of the array. Clearly $\deg_f(\alpha) = \deg_f(\beta)$. Thus each $A \in \mathrm{LDO}^0(1)$ (which contain no horizontal derivatives) decomposes to the sum $A = \sum_{m\geq 0} A_m$ with

$$A_m := \sum_{n\geq 0,\ \deg_f(\beta)=m} p_{(n,\beta)}(\mathbf{x},\mathbf{u}) \frac{\partial^n}{\partial u^n} \left(\frac{\partial}{\partial \mathbf{u}}\right)^\beta .$$

If $A \neq 0$, then the minimum $M := \min\{m;\ A_m \neq 0\}$ is defined and the supremum

$$n_M := \sup\{n;\ \text{there exists } \beta \text{ with } \deg_f(\beta) = M \text{ such that } p_{n,\beta}(\mathbf{x},\mathbf{u}) \neq 0\}$$

is, by (14), a finite number.

It is immediate to see that in the similar decomposition $E \circ A = \sum_{m\geq 0}(E\circ A)_m$ one has $(E\circ A)_m = 0$ for $m < M$ and

$$(E\circ A)_M = \sum_{\deg_f(\beta)=m} p_{(n_M,\beta)}(\mathbf{x},\mathbf{u})\frac{\partial^{n_M+1}}{\partial u^{n_M+1}}\left(\frac{\partial}{\partial \mathbf{u}}\right)^{\beta} + \text{ terms of degrees} \leq n_M \text{ in } \partial/\partial u.$$

The assumption $E\circ A = 0$ implies that $(E\circ A)_M = 0$, which in turn implies that

$$\sum_{\deg_f(\beta)=M} p_{(n_M,\beta)}\left(\frac{\partial}{\partial \mathbf{u}}\right)^{\beta} = 0,$$

which contradicts the definition of n_M.

Now in the multilinear case the operators in A which are acting on f_1 look like the terms in the case $n = 1$, involving only $(\partial/\partial u^1)^\alpha$ and no total derivatives. Essentially from the same argument isolating the terms in $E\circ A$ which have $\partial/\partial u^1$ we conclude that $A = 0$. ■

Proposition 6.4 has the following corollary.

Corollary 6.6. (‘pointwise = global’) *Given an element $A \in O^N(n)$, then $A(f_1,\ldots,f_n) \in d_H\Omega^{N-1,0}(J^\infty E)$ for each $f_1,\ldots,f_n \in \mathrm{Loc}(E)$ if and only if A is a boundary in $O^*(n)$.*

Applying the corollary, we see that (ii) implies the existence of an $\tilde{m} \in O^{N-1}(2)$ such that

$$\tilde{l}_2 + \tilde{l}_2 \circ \tau = d_H\tilde{m} \ (\tau \text{ interchanges the arguments}).$$

We may assume that the coefficients of $\tilde{m}$ are symmetric as bilinear operators, otherwise we replace $\tilde{m}$ by $\frac{1}{2}(\tilde{m} + \tilde{m}\circ\tau)$. Replacing $\tilde{l}_2$ with $\tilde{l}_2 - \frac{1}{2}d_H\tilde{m}$ gives a skew-symmetric element of $O^N(2)$ which determines the same bracket on the space of functionals $\mathcal{F}$. Similar arguments can be applied to insure appropriate skew-symmetries for all other l_k's. The next step is lifting $\tilde{l}_2$ to a chain map on the entire complex X_*. Reinterpreting condition (i) with the help of Proposition 6.4, we conclude that

$$\chi(\tilde{l}_2 \circ \frac{d}{dx^i_j}) = 0,$$

for $i = 1, 2$ and $1 \leq j \leq N$. In the notation of Theorem 4.8, $a_0(\tilde{l}_2) = 0$ and there exists a cycle $l_2 \in Z_0(\mathrm{DEnd}_*(2))$ such that $\mathcal{B}_0(l_2) = \tilde{l}_2$, that is, a chain map of LDO's lifting $\tilde{l}_2$.

Now $\mathcal{B}_0(\sum_{\sigma\in unsh(2,1)} l_2 \circ_\sigma l_2) = \sum_{\sigma\in unsh(2,1)} \tilde{l}_2 \circ_\sigma \tilde{l}_2$ and condition (iii) implies that

$$\chi(\sum_{\sigma\in unsh(2,1)} \tilde{l}_2 \circ_\sigma \tilde{l}_2) = 0.$$

The existence of $l_3 \in \mathrm{DEnd}_1(3)$ now follows from Theorem 4.8(ii). We complete the strong homotopy Lie structure using the acyclicity of $(\mathrm{DEnd}_*(n), \delta)$ in positive dimensions for all $n \geq 4$.

These results are summarized in the following theorem.

Theorem 6.7. *For $\alpha \in \Omega^{N,0}(J^\infty E)$, let $\int \alpha$ be the functional $\int \alpha[\phi] = \int (j^\infty \phi)^* \alpha$. Given $\tilde{l}_2 : \Omega^{N,0}(J^\infty E) \otimes \Omega^{N,0}(J^\infty E) \to \Omega^{N,0}(J^\infty E)$, define a bilinear map $\{-,-\} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ on the space of functionals by*

$$\{\int \alpha_1, \int \alpha_2\} := \int \tilde{l}_2(\alpha_1, \alpha_2). \tag{57}$$

If $\tilde{l}_2$ is a LDO satisfying conditions (i), (ii) and (iii) listed at the beginning of this section, then (57) defines a Lie bracket which lifts to a strong homotopy Lie algebra structure on $\Omega^{*,0}(J^\infty E)$ such that all the higher brackets are also LDOs.

In other words, all the constructions in [2] can in fact be done in the class LDO of local differential operators.

References.

[1] I.M. Anderson. Introduction to the variational bicomplex. *Cont. Math.*, 132:51–73, 1992.

[2] G. Barnich, R. Fulp, T. Lada, and J.D. Stasheff. The sh Lie structure of Poisson brackets in field theory. Preprint, 1997.

[3] I.A. Batalin and G.S. Vilkovisky. Relativistic S-matrix of dynamical systems with boson and fermion constraints. *Phys. Lett.*, pages 309–312, 1977.

[4] I.A. Batalin and G.S. Vilkovisky. Gauge algebra and quantization. *Phys. Lett.*, 102 B:27–31, 1981.

[5] I.A. Batalin and G.S. Vilkovisky. Quantization of gauge theories with linearly dependent generators. *Phys. Rev. D*, 28:2567–2582, 1983. Erratum: Phys. Rev.D 30(1984) 508.

[6] L.A. Dickey. *Soliton Equations and Hamiltonian Systems.* Advanced Series in Mathematical Physics. World Scientific, 1991.

[7] L.A. Dickey. Poisson brackets with divergence terms in field theories: two examples. Preprint, University of Oklahoma, 1997.

[8] I.M. Gel'fand and L.A. Dickey. Fractional powers of operators and hamiltonian systems. *Funkz. Anal. Priloz.*, 13(3):13–30, 1976.

[9] I.M. Gel'fand and L.A. Dickey. Lie algebra structure in the formal variational calculus. *Funkz. Anal. Priloz.*, 10(1):18–25, 1976.

[10] I.M. Gel'fand and I.Ya. Dorfman. Hamiltonian operators and associated algebraic structures. *Funkz. Anal. Priloz.*, 13(3):13–30, 1979.

[11] I.M. Gel'fand and I.Ya. Dorfman. Schouten bracket and hamiltonian operators. *Funkz. Anal. Priloz.*, 14(3):71–74, 1980.

[12] I.M. Gel'fand and I.Ya. Dorfman. Hamiltonian operators and infinite dimensional Lie algebras. *Funkz. Anal. Priloz.*, 15(3):23–40, 1981.

[13] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Communications in Algebra*, 23(6):2147–2161, 1995.

[14] M. Markl and S. Shnider. Lifting differential operator endomorphisms. Preprint, July 1998, submitted to IMRN.

[15] P.J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, 1986.

[16] V.V. Zharinov. Differential algebras and low-dimensional conservation laws. *Math. USSR Sbornik*, 71(2):319–329, 1992.

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